

Composite Wavelet Transforms: Applications and Perspectives

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ABSTRACT. We introduce a new concept of the so-called *composite wavelet transforms*. These transforms are generated by two components, namely, a kernel function and a wavelet function (or a measure). The composite wavelet transforms and the relevant Calderón-type reproducing formulas constitute a unified approach to explicit inversion of the Riesz, Bessel, Flett, parabolic and some other operators of the potential type generated by ordinary (Euclidean) and generalized (Bessel) translations. This approach is exhibited in the paper. Another concern is application of the composite wavelet transforms to explicit inversion of the k -plane Radon transform on \mathbb{R}^n . We also discuss in detail a series of open problems arising in wavelet analysis of L_p -functions of matrix argument.

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1. Introduction

Continuous wavelet transforms

$$\mathcal{W}f(x, t) = t^{-n} \int_{\mathbb{R}^n} f(y) w\left(\frac{x-y}{t}\right) dy, \quad x \in \mathbb{R}^n, \quad t > 0,$$

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where w is an integrable radial function satisfying $\int_{\mathbb{R}^n} w(x) dx = 0$, have proved to be a powerful tool in analysis and applications. There is a vast literature on this subject (see, e.g., [Da], [HO], [M], just for few). Owing to the formula

$$(1.1) \quad \int_0^\infty \mathcal{W}f(x, t) \frac{dt}{t^{1+\alpha}} = c_{\alpha, w} (-\Delta)^{\alpha/2} f(x), \quad \alpha \in \mathbb{C}, \quad \Delta = \sum_{k=1}^n \frac{\partial^2}{\partial x_k^2},$$

that can be given precise meaning, continuous wavelet transforms enable us to resolve a variety of problems dealing with powers of differential operators. Such problems arise, e.g., in potential theory, fractional calculus, and integral geometry; see, [HO], [R1]-[R7], [Tr]. Dealing with functions of several variables, it is always tempting to reduce the dimension of the domain of the wavelet function w and find new tools to gain extra flexibility. This is actually a motivation for our article.

We introduce a new concept of the so-called *composite wavelet transforms*. Loosely speaking, this is a class of wavelet-like transforms generated by two components, namely, a kernel function and a wavelet. Both are in our disposal. The first one depends on as many variables as we need for our problem. The second component, which is a wavelet function (or a measure), depends only on one variable. Such transforms are usually associated with one-parametric semigroups, like Poisson, Gauss-Weierstrass, or metaharmonic ones, and can be implemented to obtain explicit inversion formulas for diverse operators of the potential type and fractional integrals. These arise in integral geometry in a canonical way; see, e.g., [H, R2, R6, R9].

In the present article we study different types of composite wavelet transforms in the framework of the L_p -theory and the relevant Fourier and Fourier-Bessel harmonic analysis. The main focus is reproducing formulas of Calderón's type and explicit inversion of Riesz, Bessel, Flett, parabolic, and some other potentials. Apart of a brief review of recent developments in the area, the paper contains a series of new results. These include wavelet transforms for dilated kernels and wavelet transforms generated by Beta-semigroups associated to multiplication by $\exp(-t|\xi|^\beta)$, $\beta > 0$, in terms of the Fourier transform. Such semigroups arise in the context of stable random processes in probability and enjoy a number of remarkable properties [Ko], [La]. Special emphasis is made on detailed discussion of open problems arising in wavelet analysis of functions of matrix argument. Important results for L_2 -functions in this "higher-rank" set-up were obtained in [OOR] using the Fourier transform technique. The L_p -case for $p \neq 2$ is still mysterious. The main difficulties are related to correct definition and handling of admissible wavelet functions on the cone of positive definite symmetric matrices.

The paper is organized according to the Contents presented above.

2. Composite Wavelet Transforms for Dilated Kernels

2.1. Preliminaries. Let $L_p \equiv L_p(\mathbb{R}^n)$, $1 \leq p < \infty$, be the standard space of functions with the norm

$$\|f\|_p = \left(\int_{\mathbb{R}^n} |f(x)|^p dx \right)^{1/p} < \infty.$$

For technical reasons, the notation L_∞ will be used for the space $C_0 \equiv C_0(\mathbb{R}^n)$ of all continuous functions on \mathbb{R}^n vanishing at infinity. The Fourier transform of a

function f on \mathbb{R}^n is defined by

$$Ff(\xi) = \int_{\mathbb{R}^n} f(x) e^{ix \cdot \xi} dx, \quad x \cdot \xi = x_1 \xi_1 + \cdots + x_n \xi_n.$$

For $0 \leq a < b \leq \infty$, we write $\int_a^b f(\eta) d\mu(\eta)$ to denote the integral of the form $\int_{[a,b)} f(\eta) d\mu(\eta)$.

DEFINITION 2.1. Let q be a measurable function on \mathbb{R}^n satisfying the following conditions:

- (a) $q \in L_1 \cap L_r$ for some $r > 1$;
- (b) the least radial decreasing majorant of q is integrable, i.e.

$$\tilde{q}(x) = \sup_{|y| > |x|} |q(y)| \in L_1;$$

- (c) $\int_{\mathbb{R}^n} q(x) dx = 1$.

We denote

$$(2.1) \quad q_t(x) = t^{-n} q(x/t), \quad Q_t f(x) = (f * q_t)(x), \quad t > 0,$$

and set

$$(2.2) \quad Wf(x, t) = \int_0^\infty Q_{t\eta} f(x) d\mu(\eta),$$

where μ is a finite Borel measure on $[0, \infty)$. If μ is a *wavelet measure* (i.e., μ has a certain number of vanishing moments and obeys suitable decay conditions) then (2.2) will be called the *composite wavelet transform* of f . The function q will be called a *kernel function* and Q_t a *kernel operator* of the composite transform W .

The integral (2.2) is well-defined for any function $f \in L_p$, and

$$\|Wf(\cdot, t)\|_p \leq \|\mu\| \|q\|_1 \|f\|_p,$$

where $\|\mu\| = \int_{[0, \infty)} d|\mu|(\eta)$. We will also consider a more general weighted transform

$$(2.3) \quad W_a f(x, t) = \int_0^\infty Q_{t\eta} f(x) e^{-at\eta} d\mu(\eta),$$

where $a \geq 0$ is a fixed parameter.

The kernel function q , the wavelet measure μ , and the parameter $a \geq 0$ are in our disposal. This feature makes the new transform convenient in applications.

2.2. Calderón's identity. An analog of Calderón's reproducing formula for $W_a f$ is given by the following theorem.

THEOREM 2.2. Let μ be a finite Borel measure on $[0, \infty)$ satisfying

$$(2.4) \quad \mu([0, \infty)) = 0 \quad \text{and} \quad \int_0^\infty |\log \eta| d|\mu|(\eta) < \infty.$$

If $f \in L_p$, $1 \leq p \leq \infty^1$, and

$$c_\mu = \int_0^\infty \log \frac{1}{\eta} d\mu(\eta),$$

then

$$(2.5) \quad \int_0^\infty W_a f(x, t) \frac{dt}{t} \equiv \lim_{\varepsilon \rightarrow 0} \int_\varepsilon^\infty W_a f(x, t) \frac{dt}{t} = c_\mu f(x)$$

¹We remind that L_∞ is interpreted as the space C_0 with the uniform convergence.

where the limit exists in the L_p -norm and pointwise for almost all x . If $f \in C_0$, this limit is uniform on \mathbb{R}^n .

PROOF. Consider the truncated integral

$$(2.6) \quad I_\varepsilon f(x) = \int_\varepsilon^\infty W_a f(x, t) \frac{dt}{t}, \quad \varepsilon > 0.$$

Our aim is to represent it in the form

$$(2.7) \quad I_\varepsilon f(x) = \int_0^\infty Q_{\varepsilon s} f(x) e^{-a\varepsilon s} k(s) ds$$

where

$$(2.8) \quad k \in L_1(0, \infty) \quad \text{and} \quad \int_0^\infty k(s) ds = c_\mu.$$

Once (2.7) is established, all the rest follows from properties (a)-(c) in Definition 2.1 according to the standard machinery of approximation to the identity; see [St].

Equality (2.7) can be formally obtained by changing the order of integration, namely,

$$\begin{aligned} I_\varepsilon f(x) &= \int_0^\infty d\mu(\eta) \int_\varepsilon^\infty Q_{t\eta} f(x) e^{-at\eta} \frac{dt}{t} \\ &= \int_0^\infty d\mu(\eta) \int_\eta^\infty Q_{\varepsilon s} f(x) e^{-a\varepsilon s} \frac{ds}{s} \\ &= \int_0^\infty Q_{\varepsilon s} f(x) e^{-a\varepsilon s} k(s) ds, \quad k(s) = s^{-1} \int_0^s d\mu(\eta). \end{aligned}$$

Furthermore, since $\mu([0, \infty)) = 0$, then

$$\begin{aligned} \int_0^\infty |k(s)| ds &= \int_0^1 \left| \int_0^s d\mu(\eta) \right| \frac{ds}{s} + \int_1^\infty \left| \int_s^\infty d\mu(\eta) \right| \frac{ds}{s} \\ &\leq \int_0^1 d|\mu|(\eta) \int_\eta^1 \frac{ds}{s} + \int_1^\infty d|\mu|(\eta) \int_1^\eta \frac{ds}{s} \\ &= \int_0^\infty |\log \eta| d|\mu|(\eta) < \infty. \end{aligned}$$

Similarly we have

$$\int_0^\infty k(s) ds = \int_0^\infty \log \frac{1}{\eta} d\mu(\eta) = c_\mu,$$

which gives (2.8). Thus, to complete the proof, it remains to justify application of Fubini's theorem leading to (2.7). To this end, it suffices to show that the repeated integral

$$\int_\varepsilon^\infty \frac{dt}{t} \int_0^\infty |Q_{t\eta} f(x)| d|\mu|(\eta)$$

is finite for almost all x in \mathbb{R}^n . We write it as $A(x) + B(x)$, where

$$A(x) = \int_\varepsilon^\infty \frac{dt}{t} \int_0^{1/t} |Q_{t\eta} f(x)| d|\mu|(\eta), \quad B(x) = \int_\varepsilon^\infty \frac{dt}{t} \int_{1/t}^\infty |Q_{t\eta} f(x)| d|\mu|(\eta).$$

Since the least radial decreasing majorant of q is integrable (see property (b) in Definition 2.1), then $\sup_{t>0} |Q_t f(x)| \leq c M_f(x)$ where $M_f(x)$ is the Hardy-Littlewood

maximal function, which is finite for almost x ; see e.g., [St, Theorem 2, Section 2, Chapter III]. Hence, for almost x ,

$$A(x) \leq c M_f(x) \int_{\varepsilon}^{\infty} \frac{dt}{t} \int_0^{1/t} d|\mu|(\eta) = c M_f(x) \int_0^{1/\varepsilon} \left(\log \frac{1}{\eta} - \log \varepsilon \right) d|\mu|(\eta) < \infty.$$

To estimate $B(x)$, we observe that since $q \in L_r$, $r > 1$, then, by Young's inequality

$$\|Q_t f\|_s \leq \|f\|_p \|q_t\|_r = t^{-\delta} \|f\|_p \|q\|_r, \quad \delta = n(1 - 1/r) > 0, \quad \frac{1}{s} = \frac{1}{r} + \frac{1}{p} - 1.$$

This gives

$$\left\| \int_{1/t}^{\infty} |Q_{t\eta} f(x)| d|\mu|(\eta) \right\|_s \leq t^{-\delta} \|f\|_p \int_{1/t}^{\infty} \eta^{-\delta} d|\mu|(\eta),$$

and therefore,

$$\begin{aligned} \|B\|_s &\leq \|f\|_p \int_{\varepsilon}^{\infty} \frac{dt}{t^{1+\delta}} \int_{1/t}^{\infty} \eta^{-\delta} d|\mu|(\eta) \\ &= \frac{\|f\|_p}{\delta} \left(\int_0^{1/\varepsilon} d|\mu|(\eta) + \frac{1}{\varepsilon^\delta} \int_{1/\varepsilon}^{\infty} \eta^{-\delta} d|\mu|(\eta) \right) \leq \frac{\|f\|_p \|\mu\|}{\delta} < \infty. \end{aligned}$$

This completes the proof. \square

3. Wavelet Transforms Associated to One-parametric Semigroups and Inversion of Potentials

In this section we consider an important subclass of wavelet transforms, generated by certain one-parametric semigroups of operators. Some composite wavelet transforms from the previous section belong to this subclass.

3.1. Basic examples.

EXAMPLE 3.1. Consider the *Poisson semigroup* \mathcal{P}_t generated by the Poisson integral

$$(3.1) \quad \mathcal{P}_t f(x) = \int_{\mathbb{R}^n} p(y, t) f(x - y) dy, \quad t > 0$$

with the Poisson kernel

$$(3.2) \quad p(y, t) = \frac{\Gamma((n+1)/2)}{\pi^{(n+1)/2}} \frac{t}{(t^2 + |y|^2)^{(n+1)/2}} = t^{-n} p(y/t, 1);$$

see [SW2], [St]. In this specific case, the kernel function of the relevant composite wavelet transform is $q(x) \equiv p(x, 1)$ and on the Fourier transform side we have

$$(3.3) \quad F[\mathcal{P}_t f](\xi) = e^{-t|\xi|} Ff(\xi).$$

EXAMPLE 3.2. Another important example is the *Gauss-Weierstrass semigroup* \mathcal{W}_t defined by

$$(3.4) \quad \mathcal{W}_t f(x) = \int_{\mathbb{R}^n} w(y, t) f(x - y) dy, \quad F[w(\cdot, t)](\xi) = e^{-t|\xi|^2}, \quad t > 0;$$

see [SW2]. The Gauss-Weierstrass kernel $w(y, t)$ is explicitly computed as

$$(3.5) \quad w(y, t) = (4\pi t)^{-n/2} \exp(-|y|^2/4t).$$

In comparison with (2.1), here the scaling parameter t is replaced by \sqrt{t} , so that

$$(3.6) \quad w(y, t) = (\sqrt{t})^{-n} q(y/\sqrt{t}), \quad q(y) = w(y, 1) = (4\pi)^{-n/2} \exp(-|y|^2/4),$$

and the corresponding wavelet transform has the form

$$(3.7) \quad Wf(x, t) = \int_0^\infty \mathcal{W}_{t\eta} f(x) e^{-at\eta} d\mu(\eta), \quad x \in \mathbb{R}^n, \quad t > 0, \quad a \geq 0.$$

This agrees with (2.3) up to an obvious change of scaling parameters.

EXAMPLE 3.3. The following interesting example does not fall into the scope of wavelet transforms in Section 2, however, it has a very close nature. Consider the *metaharmonic semigroup* \mathcal{M}_t defined by

$$(3.8) \quad (\mathcal{M}_t f)(x) = \int_{\mathbb{R}^n} m(y, t) f(x-y) dy, \quad F[m(\cdot, t)](\xi) = e^{-t\sqrt{1+|\xi|^2}};$$

see [R1, p. 257-258]. The corresponding kernel has the form

$$(3.9) \quad m(y, t) = \frac{2t}{(2\pi)^{(n+1)/2}} \frac{K_{(n+1)/2}(\sqrt{|y|^2 + t^2})}{(\sqrt{|y|^2 + t^2})^{(n+1)/2}},$$

where $K_{(n+1)/2}(\cdot)$ is the McDonald function. The relevant wavelet transform is

$$(3.10) \quad Wf(x, t) = \int_0^\infty \mathcal{M}_{t\eta} f(x) d\mu(\eta), \quad x \in \mathbb{R}^n, \quad t > 0.$$

This list of examples can be continued [AR4].

3.2. Operators of the potential type. One of the most remarkable applications of wavelet transforms associated to the Poisson, Gauss-Weierstrass, and metaharmonic semigroups is that they pave the way to a series of explicit inversion formulas for operators of the potential type arising in analysis and mathematical physics. Typical examples of such operators are the following:

$$(3.11) \quad I^\alpha f = F^{-1} |\xi|^{-\alpha} Ff \equiv (-\Delta)^{-\alpha/2} f \quad (\text{Riesz potentials}),$$

$$(3.12) \quad J^\alpha f = F^{-1} (1 + |\xi|^2)^{-\alpha/2} Ff \equiv (E - \Delta)^{-\alpha/2} f \quad (\text{Bessel potentials}),$$

$$(3.13) \quad \mathcal{F}^\alpha f = F^{-1} (1 + |\xi|)^{-\alpha} Ff \equiv (E + \sqrt{-\Delta})^{-\alpha} f \quad (\text{Flett potentials}).$$

Here $\operatorname{Re} \alpha > 0$, $|\xi| = (\xi_1^2 + \dots + \xi_n^2)^{1/2}$, $\Delta = \sum_{k=1}^n \frac{\partial^2}{\partial x_k^2}$ is the Laplacean, and E is the identity operator. For $f \in L_p(\mathbb{R}^n)$, $1 \leq p < \infty$, these potentials have remarkable integral representations via the Poisson and Gauss-Weierstrass semigroups, namely,

$$(3.14) \quad I^\alpha f(x) = \frac{1}{\Gamma(\alpha)} \int_0^\infty t^{\alpha-1} \mathcal{P}_t f(x) dt, \quad 0 < \operatorname{Re} \alpha < n/p;$$

$$(3.15) \quad J^\alpha f(x) = \frac{1}{\Gamma(\alpha/2)} \int_0^\infty t^{\alpha/2-1} e^{-t} \mathcal{W}_t f(x) dt, \quad 0 < \operatorname{Re} \alpha < \infty;$$

$$(3.16) \quad \mathcal{F}^\alpha f(x) = \frac{1}{\Gamma(\alpha)} \int_0^\infty t^{\alpha-1} e^{-t} \mathcal{P}_t f(x) dt, \quad 0 < \operatorname{Re} \alpha < \infty;$$

see [SW1], [R1], [F1]. Regarding Flett potentials, see, in particular, [F1, p. 446-447], [SKM, p. 541-542], [ASE]. We also mention another interesting representation of the Bessel potential, which is due to Lizorkin [Li] and employs the

metaharmonic semigroup, namely,

$$(3.17) \quad J^\alpha f(x) = \frac{1}{\Gamma(\alpha)} \int_0^\infty t^{\alpha-1} \mathcal{M}_t f(x) dt, \quad 0 < \operatorname{Re} \alpha < \infty.$$

Equalities (3.14)-(3.17) have the same nature as classical Balakrishnan's formulas for fractional powers of operators (see [SKM, p. 121]).

Let us show how these equalities generate wavelet inversion formulas for the corresponding potentials. The core of the method is the following statement which is a particular case of Lemma 1.3 from [R3].

LEMMA 3.4. *Given a finite Borel measure μ on $[0, \infty)$ and a complex number α , $\alpha' = \operatorname{Re} \alpha \geq 0$, let*

$$(3.18) \quad \lambda_\alpha(s) = s^{-1} I_+^{\alpha+1} \mu(s),$$

where

$$(3.19) \quad I_+^{\alpha+1} \mu(s) = \frac{1}{\Gamma(\alpha+1)} \int_0^s (s-\eta)^\alpha d\mu(\eta)$$

is the Riemann-Liouville fractional integral of order $\alpha+1$ of the measure μ . Suppose that μ satisfies the following conditions:

$$(3.20) \quad \int_1^\infty \eta^\gamma d|\mu|(\eta) < \infty \quad \text{for some } \gamma > \alpha';$$

$$(3.21) \quad \int_0^\infty \eta^j d\mu(\eta) = 0 \quad \forall j = 0, 1, \dots, [\operatorname{Re} \alpha] \quad (\text{the integer part of } \alpha').$$

Then

$$(3.22) \quad \lambda_\alpha(s) = \begin{cases} O(s^{\alpha'-1}), & \text{if } 0 < s < 1, \\ O(s^{-1-\delta}) \text{ for some } \delta > 0, & \text{if } s > 1, \end{cases}$$

and

$$(3.23) \quad \begin{aligned} c_{\alpha, \mu} &\equiv \int_0^\infty \lambda_\alpha(s) ds = \int_0^\infty \frac{\tilde{\mu}(t)}{t^{\alpha+1}} dt \\ &= \begin{cases} \Gamma(-\alpha) \int_0^\infty \eta^\alpha d\mu(\eta) & \text{if } \alpha \notin \mathbb{N}_0 = \{0, 1, 2, \dots\}, \\ \frac{(-1)^{\alpha+1}}{\alpha!} \int_0^\infty \eta^\alpha \log \eta d\mu(\eta) & \text{if } \alpha \in \mathbb{N}_0, \end{cases} \end{aligned}$$

where $\tilde{\mu}(t) = \int_0^\infty e^{-t\eta} d\mu(\eta)$ is the Laplace transform of μ .

The estimate (3.22) is important in proving almost everywhere convergence in forthcoming inversion formulas.

Consider, for example, Flett potential (3.13), (3.16), and make use of the composite wavelet transform

$$(3.24) \quad W\varphi(x, t) = \int_0^\infty \mathcal{P}_{t\eta} \varphi(x) e^{-t\eta} d\mu(\eta),$$

cf. Example 3.1 and (2.3) with $a = 1$.

THEOREM 3.5. *Let $f \in L_p$, $1 \leq p \leq \infty$, and let $\varphi = \mathcal{F}^\alpha f$, $\alpha > 0$, be the Flett potentials of f . Suppose that μ is a finite Borel measure on $[0, \infty)$ satisfying (3.20) and (3.21). Then*

$$(3.25) \quad \int_0^\infty W_\mu \varphi(x, t) \frac{dt}{t^{1+\alpha}} \equiv \lim_{\varepsilon \rightarrow 0} \int_\varepsilon^\infty W_\mu \varphi(x, t) \frac{dt}{t^{1+\alpha}} = c_{\alpha, \mu} f(x),$$

where $c_{\alpha, \mu}$ is defined by (3.23) and the limit is interpreted in the L_p -norm and pointwise a.e. on \mathbb{R}^n . If $f \in C_0$, the statement remains true with the limit in (3.25) interpreted in the sup-norm.

PROOF. We sketch the proof and address the reader to [ASE] for details. Changing the order of integration, owing to (3.24), (3.16), and the semigroup property of the Poisson integral, we get

$$(3.26) \quad W\varphi(x, t) = \frac{1}{\Gamma(\alpha)} \int_0^\infty d\mu(\eta) \int_{t\eta}^\infty (\rho - t\eta)_+^{\alpha-1} e^{-\rho} \mathcal{P}_\rho f(x) d\rho.$$

Then further calculations give

$$(3.27) \quad \int_\varepsilon^\infty W\varphi(x, t) \frac{dt}{t^{1+\alpha}} = \int_0^\infty e^{-\varepsilon s} \mathcal{P}_{\varepsilon s} f(x) \lambda_\alpha(s) ds, \quad \lambda_\alpha(s) = s^{-1} I_+^{\alpha+1} \mu(s),$$

cf. (3.18). It remains to applied Lemma 3.4 combined with the standard machinery of approximation to the identity. \square

Potentials (3.11)-(3.13) and many others can be similarly inverted by making use of the wavelet transforms associated with suitable semigroups; see [AR4], [ASE].

3.3. Examples of wavelet measures. Examples of wavelet measures, that obey the conditions of Lemma 3.4 with $c_{\alpha, \mu} \neq 0$, are the following.

1. Fix an integer $m > \operatorname{Re} \alpha$ and choose an even Schwartz function $h(\eta)$ on \mathbb{R}^1 so that

$$h^{(k)}(0) = 0 \quad \forall k = 0, 1, 2, \dots, \quad \text{and} \quad \int_0^\infty \eta^{\alpha-m} h(\eta) d\eta \neq 0.$$

One can take, for instance, $h(\eta) = \exp(-\eta^2 - 1/\eta^2)$, $h(0) = 0$. Set $d\mu(\eta) = h^{(m)}(\eta) d\eta$. It is not difficult to show that $\int_0^\infty \eta^k d\mu(\eta) = 0$, $\forall k = 0, 1, \dots, [\operatorname{Re} \alpha]$, and $c_{\alpha, \mu} \neq 0$.

2. Let $\mu = \sum_{j=0}^m \binom{m}{j} (-1)^j \delta_j$, where $m > \operatorname{Re} \alpha$ is a fixed integer and $\delta_j = \delta_j(\eta)$

denotes the unit mass at the point $\eta = j$, i.e., $\langle \delta_j, f \rangle = f(j)$. It is known [SKM, p. 117], that

$$\int_0^\infty \eta^k d\mu(\eta) \equiv \sum_{j=0}^m \binom{m}{j} (-1)^j j^k = 0, \quad \forall k = 0, 1, \dots, m-1 \quad (\text{we set } 0^0 = 1).$$

Moreover, $c_{\alpha, \mu} = \int_0^\infty t^{-\alpha-1} (1 - e^{-t})^m dt \neq 0$.

4. Wavelet transforms with the generalized translation operator

Continuous wavelet transforms, studied in the previous sections, rely on the classical Fourier analysis on \mathbb{R}^n . Interesting modifications of these transforms and the corresponding potential operators arise in the framework of the Fourier-Bessel harmonic analysis associated to the Laplace-Bessel differential operator

$$(4.1) \quad \Delta_\nu = \sum_{k=1}^n \frac{\partial^2}{\partial x_k^2} + \frac{2\nu}{x_n} \frac{\partial}{\partial x_n}, \quad \nu > 0.$$

This analysis amounts to pioneering works by Delsarte [De] and Levitan [Le], and was extensively developed in subsequent publications; see [Ki], [Tr], [AR3], and references therein.

Let $\mathbb{R}_+^n = \{x : x = (x_1, \dots, x_n) \in \mathbb{R}^n, x_n > 0\}$ and $x' = (x_1, \dots, x_{n-1})$. Denote

$$L_{p,\nu}(\mathbb{R}_+^n) = \left\{ f : \|f\|_{p,\nu} = \left(\int_{\mathbb{R}_+^n} |f(x)|^p x_n^{2\nu} dx \right)^{1/p} < \infty \right\}.$$

The Fourier-Bessel harmonic analysis is adopted to *the generalized convolutions*

$$(4.2) \quad (f * g)(x) = \int_{\mathbb{R}_+^n} f(y) (T^y g)(x) y_n^{2\nu} dy, \quad x \in \mathbb{R}_+^n,$$

with the *generalized translation operator*

$$(4.3) \quad (T^y f)(x) = \frac{\Gamma(\nu + 1/2)}{\Gamma(\nu)\Gamma(1/2)} \int_0^\pi f(x' - y', \sqrt{x_n^2 - 2x_n y_n \cos \alpha + y_n^2}) \sin^{2\nu-1} \alpha d\alpha,$$

[Ki], [Le], [Tr]. The Fourier-Bessel transform F_ν , for which $F_\nu(f * g) = F_\nu(f) F_\nu(g)$, is defined by

$$(4.4) \quad (F_\nu f)(\xi) = \int_{\mathbb{R}_+^n} f(x) e^{i\xi' \cdot x'} j_{\nu-1/2}(\xi_n x_n) x_n^{2\nu} dx, \quad \xi \in \mathbb{R}_+^n.$$

Here $j_\lambda(\tau) = 2^\lambda \Gamma(\lambda + 1) \tau^{-\lambda} J_\lambda(\tau)$, where $J_\lambda(\tau)$ is the Bessel function of the first kind. The *generalized Gauss-Weierstrass, Poisson, and metaharmonic semigroups* $\{\mathcal{W}_t^{(\nu)}\}$, $\{\mathcal{P}_t^{(\nu)}\}$, $\{\mathcal{M}_t^{(\nu)}\}$ are defined as follows:

$$(4.5) \quad (\mathcal{W}_t^{(\nu)} f)(x) = \int_{\mathbb{R}_+^n} w^{(\nu)}(y, t) (T^y f)(x) y_n^{2\nu} dy,$$

$$(4.6) \quad (\mathcal{P}_t^{(\nu)} f)(x) = \int_{\mathbb{R}_+^n} p^{(\nu)}(y, t) (T^y f)(x) y_n^{2\nu} dy,$$

$$(4.7) \quad (\mathcal{M}_t^{(\nu)} f)(x) = \int_{\mathbb{R}_+^n} m^{(\nu)}(y, t) (T^y f)(x) y_n^{2\nu} dy,$$

$$F_\nu[w^{(\nu)}(\cdot, t)](\xi) = e^{-t|\xi|^2};$$

$$F_\nu[p^{(\nu)}(\cdot, t)](\xi) = e^{-t|\xi|};$$

$$F_\nu[m^{(\nu)}(\cdot, t)](\xi) = e^{-t\sqrt{1+|\xi|^2}}.$$

The corresponding kernels $w^{(\nu)}(y, t)$, $p^{(\nu)}(y, t)$, and $m^{(\nu)}(y, t)$ have the form

$$(4.8) \quad w^{(\nu)}(y, t) = \frac{2\pi^{\nu+1/2}}{\Gamma(\nu+1/2)} (4\pi t)^{-(n+2\nu)/2} e^{-|y|^2/4t},$$

$$(4.9) \quad p^{(\nu)}(y, t) = \frac{2\Gamma((n+2\nu+1)/2)}{\pi^{n/2}\Gamma(\nu+1/2)} \frac{t}{(|y|^2+t^2)^{(n+2\nu+1)/2}},$$

$$(4.10) \quad m^{(\nu)}(y, t) = \frac{2^{-\nu+3/2}t}{\Gamma(\nu+1/2)(2\pi)^{n/2}} \frac{K_{(n+2\nu+1)/2}(\sqrt{|y|^2+t^2})}{(\sqrt{|y|^2+t^2})^{(n+2\nu+1)/2}}.$$

More information about these semigroups and their modifications

$$\{e^{-t}\mathcal{W}_t^{(\nu)}\}, \quad \{e^{-t}\mathcal{P}_t^{(\nu)}\}, \quad \{e^{-t}\mathcal{M}_t^{(\nu)}\},$$

can be found in [AB1], [AB2], [GA].

Modified Riesz, Bessel, and Flett potentials with the generalized translation operator (4.3) are formally defined in terms of the Fourier-Bessel transform by

$$(4.11) \quad I_\nu^\alpha f = F_\nu^{-1} |\xi|^{-\alpha} F_\nu f \equiv (-\Delta_\nu)^{-\alpha/2} f,$$

$$(4.12) \quad \mathcal{J}_\nu^\alpha f = F_\nu^{-1} (1 + |\xi|^2)^{-\alpha/2} F_\nu f \equiv (E - \Delta_\nu)^{-\alpha/2} f,$$

$$(4.13) \quad \mathcal{F}_\nu^\alpha f = F_\nu^{-1} (1 + |\xi|)^{-\alpha} F_\nu f \equiv \left(E + \sqrt{-\Delta_\nu}\right)^{-\alpha} f,$$

respectively. Here $\operatorname{Re} \alpha > 0$ and Δ_ν is the Laplace-Bessel differential operator (4.1). These generalized potentials have analogous to (3.14)-(3.16) representations in terms of the semigroups (4.5)-(4.7), namely, if $f \in L_{p,\nu}(\mathbb{R}_+^n)$ then

$$(4.14) \quad I_\nu^\alpha f(x) = \frac{1}{\Gamma(\alpha)} \int_0^\infty t^{\alpha-1} \mathcal{P}_t^{(\nu)} f(x) dt, \quad 0 < \operatorname{Re} \alpha < (n+2\nu)/p,$$

$$(4.15) \quad \mathcal{J}_\nu^\alpha f(x) = \frac{1}{\Gamma(\alpha/2)} \int_0^\infty t^{\alpha/2-1} e^{-t} \mathcal{W}_t^{(\nu)} f(x) dt, \quad 0 < \operatorname{Re} \alpha < \infty,$$

$$(4.16) \quad \mathcal{F}_\nu^\alpha f(x) = \frac{1}{\Gamma(\alpha)} \int_0^\infty t^{\alpha-1} e^{-t} \mathcal{P}_t^{(\nu)} f(x) dt, \quad 0 < \operatorname{Re} \alpha < \infty.$$

Moreover,

$$(4.17) \quad \mathcal{J}_\nu^\alpha f(x) = \frac{1}{\Gamma(\alpha)} \int_0^\infty t^{\alpha-1} \mathcal{M}_t^{(\nu)} f(x) dt, \quad 0 < \operatorname{Re} \alpha < \infty.$$

We denote by $S_t^{(\nu)}$ any of the semigroups

$$(4.18) \quad \mathcal{W}_t^{(\nu)}, \quad e^{-t}\mathcal{W}_t^{(\nu)}, \quad \mathcal{P}_t^{(\nu)}, \quad e^{-t}\mathcal{P}_t^{(\nu)}, \quad \mathcal{M}_t^{(\nu)}, \quad e^{-t}\mathcal{M}_t^{(\nu)},$$

and define the relevant wavelet transform (cf. (2.2))

$$(4.19) \quad \mathfrak{S}^{(\nu)} f(x, t) = \int_0^\infty S_{t\eta}^{(\nu)} f(x) d\mu(\eta), \quad t > 0,$$

generated by a finite Borel measure μ on $[0, \infty)$.

There exist analogs of Calderón's reproducing formula for wavelet transforms (4.19) of functions belonging to the weighted space $L_{p,\nu}(\mathbb{R}_+^n)$ and inversion formulas for potentials $I_\nu^\alpha f$, $\mathcal{J}_\nu^\alpha f$, $\mathcal{F}_\nu^\alpha f$, when $f \in L_{p,\nu}(\mathbb{R}_+^n)$. For example, the following statement holds.

THEOREM 4.1. *Let $\varphi = I_\nu^\alpha f$, $f \in L_{p,\nu}(\mathbb{R}_+^n)$, $1 \leq p < (n + 2\nu)/\alpha$, and suppose that μ is a finite Borel measure on $[0, \infty)$ satisfying (3.20) and (3.21). If $\mathfrak{S}^{(\nu)}\varphi$ is the wavelet transform of φ associated with the generalized Poisson semigroup $\mathcal{P}_t^{(\nu)}$, then*

$$(4.20) \quad \int_0^\infty \frac{\mathfrak{S}^{(\nu)}\varphi(x, t)}{t^{1+\alpha}} dt = \lim_{\varepsilon \rightarrow 0} \int_\varepsilon^\infty \frac{\mathfrak{S}^{(\nu)}\varphi(x, t)}{t^{1+\alpha}} dt = c_{\alpha, \mu} f(x),$$

where $c_{\alpha, \mu}$ is defined by (3.23). The limit in (4.20) exists in the $L_{p,\nu}(\mathbb{R}_+^n)$ -norm and in the a.e. sense. If $f \in C_0$, the convergence in (4.20) is uniform.

The proof of this theorem is presented in [AR4] in the general context of the so-called admissible semigroups. This context includes all semigroups (4.18).

5. Beta-semigroups

We remind basic formulas from Section 3.1 for the kernels of the Poisson and Gauss-Weierstrass semigroups:

$$(5.1) \quad F[p(\cdot, t)](\xi) = e^{-t|\xi|}, \quad p(y, t) = \frac{\Gamma((n+1)/2)}{\pi^{(n+1)/2}} \frac{t}{(t^2 + |y|^2)^{(n+1)/2}};$$

$$(5.2) \quad F[w(\cdot, t)](\xi) = e^{-t|\xi|^2}, \quad w(y, t) = (4\pi t)^{-n/2} \exp(-|y|^2/4t).$$

It would be natural to consider a more general semigroup generated by the kernel $w^{(\beta)}(y, t)$ defined by

$$(5.3) \quad F[w^{(\beta)}(\cdot, t)](\xi) = e^{-t|\xi|^\beta}, \quad \beta > 0.$$

This semigroup arises in diverse contexts of analysis, integral geometry, and probability; see, e.g., [Fe], [Ko], [La], [R8]. Unlike (5.1) and (5.2), the kernel function $w^{(\beta)}(y, t)$ cannot be computed explicitly, however, by taking into account that

$$(5.4) \quad w^{(\beta)}(y, t) = t^{-n/\beta} w^{(\beta)}(t^{-1/\beta} y), \quad w^{(\beta)}(y) \equiv w^{(\beta)}(y, 1),$$

properties of $w^{(\beta)}(y, t)$ are well determined by the following lemma.

LEMMA 5.1. *The function*

$$(5.5) \quad w^{(\beta)}(y) = F^{-1}[e^{-|\cdot|^\beta}](y) = (2\pi)^{-n} \int_{\mathbb{R}^n} e^{-|\xi|^\beta} e^{iy \cdot \xi} d\xi, \quad \beta > 0,$$

is uniformly continuous on \mathbb{R}^n . If β is an even integer, then $w^{(\beta)}(y)$ is infinitely smooth and rapidly decreasing. More generally, if $\beta \neq 2, 4, \dots$, then $w^{(\beta)}(y)$ has the following behavior when $|y| \rightarrow \infty$:

$$(5.6) \quad w^{(\beta)}(y) = c_\beta |y|^{-n-\beta} (1 + o(|y|)), \quad c_\beta = -\frac{2^\beta \pi^{-n/2} \Gamma((n+\beta)/2)}{\Gamma(-\beta/2)}.$$

If $0 < \beta \leq 2$, then $w^{(\beta)}(y) > 0$ for all $y \in \mathbb{R}^n$.

PROOF. (Cf. [Ko, p. 44, for $n = 1$]). The uniform continuity of $w^{(\beta)}(y)$ follows immediately from (5.5). Note that if β is an even integer, then $e^{-|\cdot|^\beta}$ is a Schwartz function and therefore, $w^{(\beta)}(y)$ is infinitely smooth and rapidly decreasing. Let us prove positivity of $w^{(\beta)}(y)$ when $0 < \beta \leq 2$. For $y = 0$ and for the cases $\beta = 1$ and $\beta = 2$, this is obvious. Let $0 < \beta < 2$. By Bernstein's theorem [Fel,

Chapter 18, Sec. 4], there is a non-negative finite measure μ_β on $[0, \infty)$ so that $e^{-z^{\beta/2}} = \int_0^\infty e^{-tz} d\mu_\beta(t)$, $z \in [0, \infty)$. Replace z by $|\xi|^2$ to get

$$(5.7) \quad e^{-|\xi|^\beta} = \int_0^\infty e^{-t|\xi|^2} d\mu_\beta(t).$$

Then the equality

$$(5.8) \quad [e^{-t|\cdot|^\beta}]^\wedge(y) = \pi^{n/2} t^{-n/2} e^{-|y|^2/4t}, \quad t > 0,$$

yields

$$\begin{aligned} (2\pi)^n w^{(\beta)}(y) &= \int_{\mathbb{R}^n} e^{i\xi \cdot y} d\xi \int_0^\infty e^{-t|\xi|^2} d\mu_\beta(t) = \int_0^\infty d\mu_\beta(t) \int_{\mathbb{R}^n} e^{i\xi \cdot y} e^{-t|\xi|^2} d\xi \\ &= \pi^{n/2} \int_0^\infty t^{-n/2} e^{-|y|^2/4t} d\mu_\beta(t) > 0. \end{aligned}$$

The Fubini theorem is applicable here, because, by (5.7),

$$\int_{\mathbb{R}^n} |e^{i\xi \cdot y}| d\xi \int_0^\infty e^{-t|\xi|^2} d\mu_\beta(t) = \int_{\mathbb{R}^n} e^{-|\xi|^\beta} d\xi < \infty.$$

Let us prove (5.6). It suffices to show that

$$(5.9) \quad \lim_{|y| \rightarrow \infty} |y|^{n+\beta} w^{(\beta)}(y) = 2^\beta \pi^{-n/2-1} \Gamma(1+\beta/2) \Gamma((n+\beta)/2) \sin(\pi\beta/2)$$

(we leave to the reader to check that the right-hand side coincides with c_β). For $n = 1$, this statement can be found in [PS, Chapter 3, Problem 154] and in [Ko, p. 45]. In the general case, the proof is more sophisticated and relies on the properties of Bessel functions. By the well-known formula for the Fourier transform of a radial function (see, e.g., [SW2]), we write $(2\pi)^n w^{(\beta)}(y) = I(|\eta|)$, where

$$\begin{aligned} I(s) &= (2\pi)^{n/2} s^{1-n/2} \int_0^\infty e^{-r^\beta} r^{n/2} J_{n/2-1}(rs) dr \\ &= (2\pi)^{n/2} s^{-n} \int_0^\infty e^{-r^\beta} \frac{d}{dr} [(rs)^{n/2} J_{n/2}(rs)] dr. \end{aligned}$$

Integration by parts yields

$$I(s) = \beta(2\pi)^{n/2} s^{-n/2} \int_0^\infty e^{-r^\beta} r^{n/2+\beta-1} J_{n/2}(rs) dr.$$

Changing variable $z = s^\beta r^\beta$, we obtain

$$s^{n+\beta} I(s) = (2\pi)^{n/2} A(s^{-\beta}), \quad A(\delta) = \int_0^\infty e^{-z^\delta} z^{n/2\beta} J_{n/2}(z^{1/\beta}) dz.$$

We actually have to compute the limit $A_0 = \lim_{\delta \rightarrow 0} A(\delta)$. To this end, we invoke

Hankel functions $H_\nu^{(1)}(z)$, so that $J_\nu(z) = \operatorname{Re} H_\nu^{(1)}(z)$ if z is real [Er]. Let $h_\nu(z) = z^\nu H_\nu^{(1)}(z)$. This is a single-valued analytic function in the z -plane with cut $(-\infty, 0]$. Using the properties of the Bessel functions [Er], we get

$$(5.10) \quad \lim_{z \rightarrow 0} h_\nu(z) = 2^\nu \Gamma(\nu) / \pi i,$$

$$(5.11) \quad h_\nu(z) \sim \sqrt{2/\pi} z^{\nu-1/2} e^{iz - \frac{\pi i}{2}(\nu + \frac{1}{2})}, \quad z \rightarrow \infty.$$

Then we write $A(\delta)$ as $A(\delta) = \operatorname{Re} \int_0^\infty e^{-z^\delta} h_{n/2}(z^{1/\beta}) dz$ and change the line of integration from $[0, \infty)$ to $n_\theta = \{z : z = re^{i\theta}, r > 0\}$ for small $\theta < \pi\beta/2$. By Cauchy's

theorem, owing to (5.10) and (5.11), we obtain $A(\delta) = Re \int_{n_\theta} e^{-z^\delta} h_{n/2}(z^{1/\beta}) dz$. Since for $z = re^{i\theta}$, $h_{n/2}(z^{1/\beta}) = O(1)$ when $r = |z| \rightarrow 0$ and $h_{n/2}(z^{1/\beta}) = O(r^{(n-1)/2\beta} e^{-r^{1/\beta} \sin(\theta/\beta)})$ as $r \rightarrow \infty$, by the Lebesgue theorem on dominated convergence, we get $A_0 = Re \int_{n_\theta} h_{n/2}(z^{1/\beta}) dz$. To evaluate the last integral, we again use analyticity and replace n_θ by $n_{\pi\beta/2} = \{z : z = re^{i\pi\beta/2}, r > 0\}$ to get

$$A_0 = Re \left[e^{i\pi\beta/2} \int_0^\infty h_{n/2}(r^{1/\beta} e^{i\pi/2}) dr \right].$$

To finalize calculations, we invoke McDonald's function $K_\nu(z)$ so that

$$h_\nu(z) = z^\nu H_\nu^{(1)}(z) = -\frac{2i}{\pi} (ze^{-i\pi/2})^\nu K_\nu(ze^{-i\pi/2}).$$

This gives

$$A_0 = \frac{2\beta}{\pi} \sin(\pi\beta/2) \int_0^\infty s^{n/2+\beta-1} K_{n/2}(s) ds.$$

The last integral can be explicitly evaluated by the formula 2.16.2 (2) from [PBM], and we obtain the result. \square

The Beta-semigroup \mathcal{B}_t generated by the kernel $w^{(\beta)}(y, t)$ (see (5.3)) is defined by

$$(5.12) \quad \mathcal{B}_t f(x) = \int_{\mathbb{R}^n} w^{(\beta)}(y, t) f(x - y) dy, \quad t > 0,$$

and the corresponding weighted wavelet transform has the form

$$(5.13) \quad W_a f(x, t) = \int_0^\infty \mathcal{B}_{t\eta} f(x) e^{-at\eta} d\mu(\eta),$$

where $a \geq 0$ is a fixed number which is in our disposal; cf (2.3). Following [A1], we introduce Beta-potentials

$$(5.14) \quad J_\beta^\alpha f = (E + (-\Delta^{\beta/2}))^{-\alpha/\beta} f, \quad \alpha > 0, \quad \beta > 0,$$

that can be realized through the Beta-semigroup as

$$(5.15) \quad J_\beta^\alpha f(x) = \frac{1}{\Gamma(\alpha/\beta)} \int_0^\infty t^{\alpha/\beta-1} e^{-t} \mathcal{B}_t f(x) dt.$$

For $\beta = 2$, (5.14) coincides with the classical Bessel potential (3.12), and (5.15) mimics (3.15). Similarly, for $\beta = 1$, the Beta-potentials coincide with the Flett potential (3.16).

Explicit inversion formulas for Beta-potentials can be obtained with the aid of the wavelet transform (5.13) as follows.

THEOREM 5.2. *Let $f \in L_p(\mathbb{R}^n)$, $1 \leq p < \infty$, $\alpha > 0$, $\beta > 0$. Suppose that μ is a finite Borel measure on $[0, \infty)$ satisfying*

$$(a) \quad \int_1^\infty \eta^\gamma d|\mu|(\eta) < \infty \quad \text{for some } \gamma > \alpha/\beta;$$

$$(b) \quad \int_0^\infty \eta^j d\mu(\eta) = 0, \quad \forall j = 0, 1, \dots, [\alpha/\beta].$$

If $\varphi = J_\beta^\alpha f$, then

$$(5.16) \quad \int_0^\infty W\varphi(x, t) \frac{dt}{t^{1+\alpha/\beta}} \equiv \lim_{\varepsilon \rightarrow 0} \int_\varepsilon^\infty W\varphi(x, t) \frac{dt}{t^{1+\alpha/\beta}} = c_{\alpha/\beta, \mu} f(x),$$

where $c_{\alpha/\beta, \mu}$ is defined by (3.23) (with α replaced by α/β). The limit in (5.17) exists in the L_p -norm and pointwise for almost all x . If $f \in C_0$, the convergence is uniform.

The proof of this theorem mimics that of Theorem 3.5; see [A1] for details.

REMARK 5.3. The classical Riesz potential $I^\alpha f$ has an integral representation via the Beta-semigroup, namely,

$$(5.17) \quad I^\alpha f(x) = \frac{1}{\Gamma(\alpha/\beta)} \int_0^\infty t^{\alpha/\beta-1} \mathcal{B}_t f(x) dt.$$

Here $f \in L_p(\mathbb{R}^n)$, $1 \leq p < \infty$, and $0 < \operatorname{Re} \alpha < n/p$. For the cases $\beta = 1$ and $\beta = 2$ we have the representations in terms of the Poisson and Gauss-Weierstrass semigroups, respectively.

The potential $I^\alpha f$ can be inverted in the framework of the L_p -theory by making use of (5.17) and the composite wavelet transform (5.13) with $a = 0$.

6. Parabolic Wavelet Transforms

The following anisotropic wavelet transforms of the composite type, associated with the heat operators

$$(6.1) \quad \partial/\partial t - \Delta, \quad E + \partial/\partial t - \Delta,$$

were introduced by Aliev and Rubin [AR2]. These transforms are constructed using the Gauss-Weierstrass kernel $w(y, t) = (4\pi t)^{-n/2} \exp(-|y|^2/4t)$ as follows. Let \mathbb{R}^{n+1} be the $(n+1)$ -dimensional Euclidean space of points (x, t) , $x = (x_1, \dots, x_n) \in \mathbb{R}^n$, $t \in \mathbb{R}^1$. We pick up a wavelet measure μ on $[0, \infty)$, a scaling parameter $a > 0$, and set

$$(6.2) \quad P_\mu f(x, t; a) = \int_{\mathbb{R}^n \times (0, \infty)} f(x - \sqrt{a}y, t - a\tau) w(y, \tau) dy d\mu(\tau),$$

$$(6.3) \quad \mathcal{P}_\mu f(x, t; a) = \int_{\mathbb{R}^n \times (0, \infty)} f(x - \sqrt{a}y, t - a\tau) w(y, \tau) e^{-a\tau} dy d\mu(\tau)$$

(to simplify the notation, without loss of generality we can assume $\mu(\{0\}) = 0$). We call (6.2) and (6.3) the *parabolic wavelet transform* and the *weighted parabolic wavelet transform*, respectively.

Parabolic potentials $H^\alpha f$ and $\mathcal{H}^\alpha f$, associated to differential operators in (6.1), are defined in the Fourier terms by

$$(6.4) \quad F[H^\alpha f](\xi, \tau) = (|\xi|^2 + i\tau)^{-\alpha/2} F[f](\xi, \tau),$$

$$(6.5) \quad F[\mathcal{H}^\alpha f](\xi, \tau) = (1 + |\xi|^2 + i\tau)^{-\alpha/2} F[f](\xi, \tau),$$

where F stands for the Fourier transform in \mathbb{R}^{n+1} . These potentials were introduced by Jones [Jo] and Sampson [Sa] and used as a tool for characterization of anisotropic function spaces of fractional smoothness; see [AR2] and references therein. For $\alpha > 0$, potentials $H^\alpha f$ and $\mathcal{H}^\alpha f$ are representable by the integrals

$$(6.6) \quad H^\alpha f(x, t) = \frac{1}{\Gamma(\alpha/2)} \int_{\mathbb{R}^n \times (0, \infty)} \tau^{\alpha/2-1} w(y, \tau) f(x - y, t - \tau) dy d\tau,$$

$$(6.7) \quad \mathcal{H}^\alpha f(x, t) = \frac{1}{\Gamma(\alpha/2)} \int_{\mathbb{R}^n \times (0, \infty)} \tau^{\alpha/2-1} e^{-\tau} w(y, \tau) f(x - y, t - \tau) dy d\tau.$$

Their behavior on functions $f \in L_p \equiv L_p(\mathbb{R}^{n+1})$ is characterized by the following theorem.

THEOREM 6.1. [Ba], [Ra]

- I. Let $f \in L_p$, $1 \leq p < \infty$, $0 < \alpha < (n+2)/p$, $q = (n+2-\alpha p)^{-1}(n+2)p$.
- (a) The integral $(H^\alpha f)(x, t)$ converges absolutely for almost all $(x, t) \in \mathbb{R}^{n+1}$.
 - (b) For $p > 1$, the operator H^α is bounded from L_p into L_q .
 - (c) For $p = 1$, H^α is an operator of the weak $(1, q)$ type:

$$|\{(x, t) : |(H^\alpha f)(x, t)| > \gamma\}| \leq \left(\frac{c\|f\|_1}{\gamma} \right)^q.$$

- II. The operator \mathcal{H}^α is bounded on L_p for all $\alpha \geq 0$, $1 \leq p \leq \infty$.

Explicit inversion formulas for parabolic potentials in terms of wavelet transforms (6.2) and (6.3) are given by the following theorem.

THEOREM 6.2. [AR2] Let μ be a finite Borel measure on $[0, \infty)$ satisfying the following conditions:

$$(6.8) \quad \int_1^\infty \tau^\gamma d|\mu|(t) < \infty \quad \text{for some } \gamma > \alpha/2;$$

$$(6.9) \quad \int_0^\infty t^j d\mu(t) = 0, \quad \forall j = 0, 1, \dots, [\alpha/2].$$

Suppose that $\varphi = H^\alpha f$, $f \in L_p$, $1 \leq p < \infty$, $0 < \alpha < (n+2)/p$. Then

$$(6.10) \quad \int_0^\infty P_\mu \varphi(x, t; a) \frac{da}{a^{1+\alpha/2}} \equiv \lim_{\varepsilon \rightarrow 0} \int_\varepsilon^\infty (\dots) = c_{\alpha/2, \mu} f(x, t),$$

where $c_{\alpha/2, \mu}$ is defined by (3.23) (with α replaced by $\alpha/2$).

The limit in (6.10) is interpreted in the L_p -norm for $1 \leq p < \infty$ and a.e. on \mathbb{R}^{n+1} for $1 < p < \infty$.

The same statement holds for all $\alpha > 0$ and $1 \leq p \leq \infty$ (L_∞ is identified with C_0) provided that H^α and P_μ are replaced by \mathcal{H}^α and \mathcal{P}_μ , respectively.

More general results for parabolic wavelet transforms with the generalized translation associated to singular heat operators

$$(6.11) \quad \partial/\partial t - \Delta_\nu, \quad E + \partial/\partial t - \Delta_\nu, \quad \left(\Delta_\nu = \sum_{k=1}^n \frac{\partial^2}{\partial x_k^2} + \frac{2\nu}{x_n} \frac{\partial}{\partial x_n} \right),$$

were obtained in [AR1]. These include the Calderón-type reproducing formula and explicit L_p -inversion formulas for parabolic potentials with the generalized translation defined by

$$\begin{aligned} H_\nu^\alpha f(x, t) &= F_\nu^{-1}[(|x|^2 + it)^{-\alpha/2} F_\nu f(x, t)], \\ \mathcal{H}_\nu^\alpha f(x, t) &= F_\nu^{-1}[(1 + |x|^2 + it)^{-\alpha/2} F_\nu f(x, t)]. \end{aligned}$$

In the last two expressions, $x \in \mathbb{R}_+^n = \{x \in \mathbb{R}^n : x_n > 0\}$, $t \in \mathbb{R}^1$, and F_ν is the Fourier-Bessel transform, i.e., the Fourier transform with respect to the variables t and $x' = (x_1, \dots, x_{n-1})$, and the Bessel transform with respect to $x_n > 0$. These results were applied in [AR1, AR2] to wavelet-type characterization of the parabolic Lebesgue spaces.

7. Some Applications to Inversion of the k -plane Radon Transform

We recall some basic definitions. More information can be found in [GGG, H, E, R4, R5]. Let $\mathcal{G}_{n,k}$ and $G_{n,k}$ be the affine Grassmann manifold of all non-oriented k -dimensional planes (k -planes) τ in \mathbb{R}^n and the ordinary Grassmann manifold of k -dimensional linear subspaces ζ of \mathbb{R}^n , respectively. Each k -plane $\tau \in \mathcal{G}_{n,k}$ is parameterized as $\tau = (\zeta, u)$, where $\zeta \in G_{n,k}$ and $u \in \zeta^\perp$ (the orthogonal complement of ζ in \mathbb{R}^n). We endow $\mathcal{G}_{n,k}$ with the product measure $d\tau = d\zeta du$, where $d\zeta$ is the $O(n)$ -invariant measure on $G_{n,k}$ of total mass 1, and du denotes the Euclidean volume element on ζ^\perp . The k -plane Radon transform of a function f on \mathbb{R}^n is defined by

$$(7.1) \quad \hat{f}(\tau) \equiv \hat{f}(\zeta, u) = \int_{\zeta} f(y + u) dy,$$

where dy is the induced Lebesgue measure on the subspace $\zeta \in G_{n,k}$. This transform assigns to a function f a collection of integrals of f over all k -planes in \mathbb{R}^n . The corresponding *dual k -plane transform* of a function φ on $\mathcal{G}_{n,k}$ is defined as the mean value of $\varphi(\tau)$ over all k -planes τ through $x \in \mathbb{R}^n$:

$$(7.2) \quad \check{\varphi}(x) = \int_{O(n)} \varphi(\sigma \zeta_0 + x) d\sigma, \quad x \in \mathbb{R}^n.$$

Here $\zeta_0 \in G_{n,k}$ is an arbitrary fixed k -plane through the origin. If $f \in L_p(\mathbb{R}^n)$, then \hat{f} is finite a.e. on $\mathcal{G}_{n,k}$ if and only if $1 \leq p < n/k$.

Several inversion procedures are known for \hat{f} . One of the most popular, which amounts to Blaschke and Radon, relies on the Fuglede formula [H, p. 29],

$$(7.3) \quad (\hat{f})^\vee = d_{k,n} I^k f, \quad d_{k,n} = (2\pi)^k \sigma_{n-k-1} / \sigma_{n-1},$$

and reduces reconstruction of f to inversion of the Riesz potentials $I^k f$. The latter can also be inverted in many number of ways [S], [SKM], [R1]. In view of considerations in Section 3.2 and 5, one can employ a composite wavelet transform generated by the Poisson, Gauss-Weierstrass, or Beta semigroup and thus obtain new inversion formulas for the k -plane transform on \mathbb{R}^n in terms of a wavelet measure on the one-dimensional set $[0, \infty)$. For instance, this way leads to the following

THEOREM 7.1. *Let $\varphi = \hat{f}$ be the k -plane Radon transform of a function $f \in L_p$, $1 \leq p < n/k$. Let μ be a finite Borel measure on $[0, \infty)$ satisfying*

$$(a) \quad \int_1^\infty \eta^\gamma d|\mu|(\eta) < \infty \quad \text{for some } \gamma > k;$$

$$(b) \quad \int_0^\infty \eta^j d\mu(\eta) = 0 \quad \forall j = 0, 1, \dots, k.$$

Let $W\check{\varphi}$ be the wavelet transform of $\check{\varphi}$, associated with the Poisson semigroup (3.1), namely,

$$(7.4) \quad W\check{\varphi}(x, t) = \int_0^\infty \mathcal{P}_{t\eta} \check{\varphi}(x) d\mu(\eta), \quad x \in \mathbb{R}^n, \quad t > 0.$$

Then

$$(7.5) \quad \int_0^\infty W\check{\varphi}(x, t) \frac{dt}{t^{1+k}} \equiv \lim_{\varepsilon \rightarrow \infty} \int_\varepsilon^\infty W\check{\varphi}(x, t) \frac{dt}{t^{1+k}} = c_{k,\mu} f(x),$$

where (cf. (3.23)),

$$c_{k,\mu} = \frac{(-1)^{k+1}}{k!} \int_0^\infty t^k \log t \, d\mu(t).$$

The limit in (7.5) exists in the L_p -norm and pointwise almost everywhere. If $f \in C_0 \cap L_p$, the convergence is uniform on \mathbb{R}^n .

REMARK 7.2. The following observation might be interesting. Let

$$(7.6) \quad I_-^\alpha u(t) = \frac{1}{\Gamma(\alpha)} \int_t^\infty (t-s)^{\alpha-1} u(s) \, ds, \quad t > 0,$$

be the Riemann-Liouville integral of u . It is known [R1, formula (16.9)] that the Poisson integral takes the Riesz potential $I^\alpha f$ to the Riemann-Liouville integral of the function $t \rightarrow \mathcal{P}_t f$, namely,

$$(7.7) \quad \mathcal{P}_t I^\alpha f = I_-^\alpha \mathcal{P}_t f.$$

Denoting by R and R^* the Radon k -plane transform and its dual, owing to Fuglede's formula (7.3), we have

$$(7.8) \quad R^* R f = d_{k,n} I^k f.$$

Combining (7.8) and (7.7), we get

$$(7.9) \quad R_t^* R f = d_{k,n} I_-^k \mathcal{P}_t f, \quad R_t^* \varphi(x) = (\mathcal{P}_t R^* \varphi)(x).$$

This formula has the same nature as the following one in terms of the spherical means, that lies in the scope of the classical Funk-Radon-Helgason theory:

$$(7.10) \quad (\hat{f})_r^\vee(x) = \sigma_{k-1} \int_r^\infty (\mathfrak{M}_t f)(x) (t^2 - r^2)^{k/2-1} t \, dt;$$

see Lemma 5.1 in [R4]. Here σ_{k-1} is the volume of the $(k-1)$ -dimensional unit sphere,

$$(7.11) \quad (\mathfrak{M}_t f)(x) = \frac{1}{\sigma_{n-1}} \int_{S^{n-1}} f(x + t\theta) \, d\theta, \quad t > 0,$$

and $(\hat{f})_r^\vee(x)$ is the so-called *shifted dual k -plane transform*, which is the mean value of $\hat{f}(\tau)$ over all k -planes τ at distance r from x .

8. Higher-rank Composite Wavelet Transforms and Open Problems

Challenging perspectives and open problems for composite wavelet transforms are connected with functions of matrix argument and their application to integral geometry. This relatively new area encompasses the so-called higher-rank problems, when traditional scalar notions, like distance or scaling, become matrix-valued.

8.1. Matrix spaces, preliminaries. We remind basic notions, following [R9]. Let $\mathfrak{M}_{n,m} \sim \mathbb{R}^{nm}$ be the space of real matrices $x = (x_{i,j})$ having n rows and m columns, $n \geq m$; $dx = \prod_{i=1}^n \prod_{j=1}^m dx_{i,j}$ is the volume element on $\mathfrak{M}_{n,m}$, x' denotes the transpose of x , and I_m is the identity $m \times m$ matrix. Given a square matrix a , we denote by $\det(a)$ the determinant of a , and by $|a|$ the absolute value of $\det(a)$; $\text{tr}(a)$ stands for the trace of a . For $x \in \mathfrak{M}_{n,m}$, we denote

$$(8.1) \quad |x|_m = \det(x'x)^{1/2}.$$

If $m = 1$, this is the usual Euclidean norm on \mathbb{R}^n . For $m > 1$, $|x|_m$ is the volume of the parallelepiped spanned by the column-vectors of x . We use standard notations

$O(n)$ and $SO(n)$ for the orthogonal group and the special orthogonal group of \mathbb{R}^n with the normalized invariant measure of total mass 1. Let $\mathcal{S}_m \sim \mathbb{R}^{m(m+1)/2}$ be the space of $m \times m$ real symmetric matrices $s = (s_{i,j})$ with the volume element $ds = \prod_{i \leq j} ds_{i,j}$. We denote by \mathcal{P}_m the cone of positive definite matrices in \mathcal{S}_m ; $\overline{\mathcal{P}}_m$ is the closure of \mathcal{P}_m , that is, the set of all positive semi-definite $m \times m$ matrices. For $r \in \mathcal{P}_m$ ($r \in \overline{\mathcal{P}}_m$), we write $r > 0$ ($r \geq 0$). Given a and b in \mathcal{S}_m , the inequality $a > b$ means $a - b \in \mathcal{P}_m$ and the symbol $\int_a^b f(s)ds$ denotes the integral over the set $(a + \mathcal{P}_m) \cap (b - \mathcal{P}_m)$.

The group $G = GL(m, \mathbb{R})$ of real non-singular $m \times m$ matrices g acts transitively on \mathcal{P}_m by the rule $r \rightarrow grg'$. The corresponding G -invariant measure on \mathcal{P}_m is

$$(8.2) \quad d_*r = |r|^{-d}dr, \quad |r| = \det(r), \quad d = (m+1)/2$$

[Te, p. 18].

LEMMA 8.1. [Mu, pp. 57–59]

- (i) If $x = ayb$ where $y \in \mathfrak{M}_{n,m}$, $a \in GL(n, \mathbb{R})$, and $b \in GL(m, \mathbb{R})$, then $dx = |a|^m |b|^n dy$.
- (ii) If $r = q'sq$ where $s \in \mathcal{S}_m$, and $q \in GL(m, \mathbb{R})$, then $dr = |q|^{m+1} ds$.
- (iii) If $r = s^{-1}$ where $s \in \mathcal{P}_m$, then $r \in \mathcal{P}_m$, and $dr = |s|^{-m-1} ds$.

For $Re \alpha > d - 1$, the Siegel gamma function of \mathcal{P}_m is defined by

$$(8.3) \quad \Gamma_m(\alpha) = \int_{\mathcal{P}_m} \exp(-\text{tr}(r)) |r|^\alpha d_*r = \pi^{m(m-1)/4} \prod_{j=0}^{m-1} \Gamma(\alpha - j/2),$$

[FK, Te]. The relevant beta function has the form

$$(8.4) \quad B_m(\alpha, \beta) = \int_0^{I_m} |r|^{\alpha-d} |I_m - r|^{\beta-d} dr = \frac{\Gamma_m(\alpha) \Gamma_m(\beta)}{\Gamma_m(\alpha + \beta)}, \quad d = (m+1)/2.$$

This integral converges absolutely if and only if $Re \alpha, Re \beta > d - 1$.

All function spaces on $\mathfrak{M}_{n,m}$ are identified with the corresponding spaces on \mathbb{R}^{nm} . For instance, $\mathcal{S}(\mathfrak{M}_{n,m})$ denotes the Schwartz space of infinitely differentiable rapidly decreasing functions. The Fourier transform of a function $f \in L_1(\mathfrak{M}_{n,m})$ is defined by

$$(8.5) \quad \mathcal{F}f(y) = \int_{\mathfrak{M}_{n,m}} \exp(\text{tr}(iy'x)) f(x) dx, \quad y \in \mathfrak{M}_{n,m}.$$

The Cayley-Laplace operator Δ on $\mathfrak{M}_{n,m}$ is defined by

$$(8.6) \quad \Delta = \det(\partial' \partial), \quad \partial = (\partial / \partial x_{i,j}).$$

In terms of the Fourier transform, the action of Δ represents a multiplication by the homogeneous polynomial $(-1)^m |y|_m^2$ of degree $2m$ of nm variables $y_{i,j}$.

For the sake of simplicity, for some operators on functions of matrix argument we will use the same notation as in the previous sections.

The Gårding-Gindikin integrals of functions f on \mathcal{P}_m are defined by

$$(8.7) \quad (I_+^\alpha f)(s) = \frac{1}{\Gamma_m(\alpha)} \int_0^s f(r) |s-r|^{\alpha-d} dr, \quad (I_-^\alpha f)(s) = \frac{1}{\Gamma_m(\alpha)} \int_s^\infty f(r) |r-s|^{\alpha-d} dr,$$

where $s \in \mathcal{P}_m$ in the first integral and $s \in \overline{\mathcal{P}}_m$ in the second one. We assume $Re \alpha > d - 1$, $d = (m+1)/2$ (this condition is necessary for absolute convergence

of these integrals). The first integral exists a.e. for arbitrary locally integrable function f . Existence of the second integral requires extra assumptions for f at infinity.

The *Riesz potential* of a function $f \in \mathcal{S}(\mathfrak{M}_{n,m})$ is defined by

$$(8.8) \quad (I^\alpha f)(x) = \frac{1}{\gamma_{n,m}(\alpha)} \int_{\mathfrak{M}_{n,m}} f(x-y) |y|_m^{\alpha-n} dy;$$

$$(8.9) \quad \gamma_{n,m}(\alpha) = \frac{2^{\alpha m} \pi^{nm/2} \Gamma_m(\alpha/2)}{\Gamma_m((n-\alpha)/2)}, \quad \text{Re } \alpha > m-1, \quad \alpha \neq n-m+1, n-m+2, \dots$$

This integral is finite a.e. for $f \in L_p(\mathfrak{M}_{n,m})$ provided $1 \leq p < n(\text{Re } \alpha + m - 1)^{-1}$ [R9, Theorem 5.10].

An application of the Fourier transform gives

$$(8.10) \quad \mathcal{F}[I^\alpha f](\xi) = |\xi|_m^{-\alpha} \mathcal{F}f(\xi)$$

(as in the case of \mathbb{R}^n), so that I^α can be formally identified with the negative power of the Cayley-Laplace operator (8.6), namely, $I^\alpha = (-\Delta_m)^{-\alpha/2}$. Discussion of precise meaning of the equality (8.10) and related references can be found in [R9], [OR2].

DEFINITION 8.2. For $x \in \mathfrak{M}_{n,m}$, $n \geq m$, and $t \in \mathcal{P}_m$, we define the (generalized) heat kernel $h_t(x)$ by the formula

$$(8.11) \quad h_t(x) = (4\pi)^{-nm/2} |t|^{-n/2} \exp(-\text{tr}(t^{-1}x'x)/4), \quad |t| = \det(t),$$

and set

$$(8.12) \quad H_t f(x) = \int_{\mathfrak{M}_{n,m}} h_t(x-y) f(y) dy = \int_{\mathfrak{M}_{n,m}} h_{I_m}(y) f(x - yt^{1/2}) dy.$$

Clearly, $H_t f(x)$ is a generalization of the Gauss-Weierstrass integral (3.4).

LEMMA 8.3. [R9]

(i) For each $t \in \mathcal{P}_m$,

$$(8.13) \quad \int_{\mathfrak{M}_{n,m}} h_t(x) dx = 1.$$

(ii) The Fourier transform of $h_t(x)$ has the form

$$(8.14) \quad \mathcal{F}h_t(y) = \exp(-\text{tr}(ty'y)),$$

which implies the semi-group property

$$(8.15) \quad h_t * h_\tau = h_{t+\tau}, \quad t, \tau \in \mathcal{P}_m.$$

(iii) If $f \in L_p(\mathfrak{M}_{n,m})$, $1 \leq p \leq \infty$, then

$$(8.16) \quad \|H_t f\|_p \leq \|f\|_p, \quad H_t H_\tau f = H_{t+\tau} f,$$

and

$$(8.17) \quad \lim_{t \rightarrow 0} (H_t f)(x) = f(x)$$

in the L_p -norm. If f is a continuous function vanishing at infinity, then (8.17) holds in the sup-norm.

THEOREM 8.4. **[R9]** Let $m - 1 < \operatorname{Re} \alpha < n - m + 1$, $d = (m + 1)/2$. Then

$$(8.18) \quad (I^\alpha f)(x) = \frac{1}{\Gamma_m(\alpha/2)} \int_{\mathcal{P}_m} |t|^{\alpha/2} H_t f(x) d_* t, \quad d_* t = |t|^{-d} dt,$$

$$(8.19) \quad H_t[I^\alpha f](x) = I_-^{\alpha/2}[H_{(\cdot)} f(x)](t),$$

provided that integrals on either side of the corresponding equality exist in the Lebesgue sense.

8.2. Composite wavelet transforms: open problems. Formula (8.18) provokes a natural construction of the relevant composite wavelet transform on $\mathfrak{M}_{n,m}$ associated with the heat kernel and containing a \mathcal{P}_m -valued scaling parameter. To find this construction, we first obtain an auxiliary integral representation of a power function of the form $|t|^{\lambda-d}$, $d = (m + 1)/2$.

DEFINITION 8.5. A function w on \mathcal{P}_m is said to be symmetric if

$$(8.20) \quad w(g\eta g^{-1}) = w(\eta) \quad \text{for all } g \in GL(m, \mathbb{R}), \quad \eta \in \mathcal{P}_m.$$

Note that if w is symmetric, then for any $s, t \in \mathcal{P}_m$,

$$(8.21) \quad w(t^{1/2} s t^{1/2}) = w(s^{1/2} t s^{1/2}) \quad \text{and} \quad w(ts) = w(st).$$

Indeed, the second equality follows from (8.20) if we set $\eta = ts$, $g = t^{-1}$. The first equality in (8.21) is a consequence of the second one:

$$w(t^{1/2} s t^{1/2}) = w(t^{-1/2} [t^{1/2} s t^{1/2}] t^{1/2}) = w(st) = w(ts) = w(s^{1/2} t s^{1/2}).$$

LEMMA 8.6. Let w be a symmetric function on \mathcal{P}_m satisfying

$$(8.22) \quad \int_{\mathcal{P}_m} \frac{|w(\eta)|}{|\eta|^\lambda} d\eta < \infty, \quad c = \int_{\mathcal{P}_m} \frac{w(\eta)}{|\eta|^\lambda} d\eta \neq 0, \quad |\eta| = \det(\eta).$$

Then for $t \in \mathcal{P}_m$,

$$(8.23) \quad |t|^{\lambda-d} = c^{-1} \int_{\mathcal{P}_m} \frac{w(a^{-1}t)}{|a|^{m+1-\lambda}} da, \quad d = (m + 1)/2.$$

PROOF. By (8.21) we have (set $a = \rho^{-1}$, $da = \rho^{-2d} d\rho$)

$$\begin{aligned} \int_{\mathcal{P}_m} \frac{w(a^{-1}t)}{|a|^{m+1-\lambda}} da &= \int_{\mathcal{P}_m} \frac{w(t^{1/2} a^{-1} t^{1/2})}{|a|^{m+1-\lambda}} da = \int_{\mathcal{P}_m} \frac{w(t^{1/2} \rho t^{1/2})}{|\rho|^{\lambda-d}} d_* \rho \\ &= |t|^{\lambda-d} \int_{\mathcal{P}_m} \frac{w(\eta)}{|\eta|^\lambda} d\eta = c |t|^{\lambda-d}. \end{aligned}$$

□

Now we replace a power function in (8.18) according to (8.23) with $\lambda = \alpha/2$. For $\operatorname{Re} \alpha > (m - 1)/2$, we obtain

$$\begin{aligned} (I^\alpha f)(x) &= \frac{c^{-1}}{\Gamma_m(\alpha/2)} \int_{\mathcal{P}_m} H_t f(x) dt \int_{\mathcal{P}_m} \frac{w(a^{-1}t)}{|a|^{m+1-\alpha/2}} da \\ &= \frac{c^{-1}}{\Gamma_m(\alpha/2)} \int_{\mathcal{P}_m} \frac{d_* a}{|a|^{d-\alpha/2}} \int_{\mathcal{P}_m} H_t f(x) w(a^{-1}t) dt. \end{aligned}$$

This gives

$$(8.24) \quad (I^\alpha f)(x) = \frac{c^{-1}}{\Gamma_m(\alpha/2)} \int_{\mathcal{P}_m} \mathcal{H}f(x, a) |a|^{\alpha/2} d_* a, \quad \operatorname{Re} \alpha > (m - 1)/2,$$

with

$$(8.25) \quad \mathcal{H}f(x, a) = |a|^{-d} \int_{\mathcal{P}_m} H_t f(x) w(a^{-1}t) dt$$

or, by the symmetry of w , after changing variable,

$$(8.26) \quad \mathcal{H}f(x, a) = \int_{\mathcal{P}_m} H_{a^{1/2}\eta a^{1/2}} f(x) w(\eta) d\eta, \quad x \in \mathfrak{M}_{n,m}, \quad a \in \mathcal{P}_m.$$

Taking into account an obvious similarity between (8.26) and the corresponding “rank-one” formula for $m = 1$, we call $\mathcal{H}f(x, a)$ the *composite wavelet transform of f associated to the heat semigroup H_t* . Here w is a symmetric integrable function on \mathcal{P}_m (that will be endowed later with some cancelation properties) and a is a \mathcal{P}_m -valued scaling parameter. One can replace w by a more general *wavelet measure*, as we did in the previous sections, but here we want to minimize technicalities.

Owing to (8.10), it is natural to expect that the inverse of I^α has the same form (8.24) with α formally replaced by $-\alpha$, and the case $\alpha = 0$ gives a variant of Calderón’s reproducing formula.

Thus, we encounter the following open problem:

Problem A. *Give precise meaning to the inversion formula*

$$(8.27) \quad f(x) = c_{m,\alpha} \int_{\mathcal{P}_m} \frac{\mathcal{H}\varphi(x, a)}{|a|^{\alpha/2}} d_*a, \quad \varphi = I^\alpha f,$$

and the reproducing formula

$$(8.28) \quad f(x) = c_m \int_{\mathcal{P}_m} \mathcal{H}f(x, a) d_*a,$$

say, for $f \in L_p$ or any other “natural” function space. Give examples of wavelet functions w for which (8.27) and (8.28) hold. Find explicit formulas for the normalizing coefficients $c_{m,\alpha}$ and c_m , depending on w .

Solution of this problem would give a series of pointwise inversion formulas for diverse Radon-like transforms on matrix spaces; see, e.g., [OR2], [OR3], [R9], where such formulas are available in terms of distributions. Justification of (8.27) and (8.28) would also bring new light to a variety of inversion formulas for Radon transforms on Grassmannians, cf. [GRu].

8.3. Some discussion. Trying to solve Problem A, we come across new problems that are of independent interest. Let $\operatorname{Re} \alpha > d - 1$, $d = (m + 1)/2$. Suppose, for instance, that $f(x)$, $x \in \mathfrak{M}_{n,m}$, is a Schwartz function and $w(\eta)$, $\eta \in \mathcal{P}_m$, is “good enough”. We anticipate the following equality:

$$(8.29) \quad I_\varepsilon f(x) \equiv \int_{\varepsilon I_m}^\infty \frac{\mathcal{H}[I^\alpha f](x, a)}{|a|^{\alpha/2}} d_*a = \int_{\mathcal{P}_m} \Lambda_{\alpha/2}(s) H_{\varepsilon s} f(x) ds,$$

where $\Lambda_{\alpha/2}(s)$ expresses through the Gårding-Gindikin integral in (8.7) as

$$(8.30) \quad \Lambda_{\alpha/2}(s) = \frac{\Gamma_m(d)}{|s|^d} I_+^{\alpha/2+d} w(s), \quad s \in \mathcal{P}_m.$$

If $m = 1$ and $\alpha/2$ is replaced by α , then (8.30) coincides with the function $\lambda_\alpha(s) = s^{-1} I_+^{\alpha+1} \mu(s)$ in Lemma 3.4. Now, we give the following

DEFINITION 8.7. An integrable symmetric function w on \mathcal{P}_m is called an *admissible wavelet* if

$$(8.31) \quad \Lambda_{\alpha/2}(s) \equiv \frac{\Gamma_m(d)}{|s|^d} I_+^{\alpha/2+d} w(s) \in L_1(\mathcal{P}_m) \quad \text{and} \quad c_\alpha = \int_{\mathcal{P}_m} \Lambda_{\alpha/2}(s) ds \neq 0.$$

If w is admissible, then, by Lemma 8.3, the L_p -limit as $\varepsilon \rightarrow 0$ of the right-hand side of (8.29) is $c_\alpha f$, and we are done. This discussion includes the case $\alpha = 0$ corresponding to the reproducing formula.

Thus, our attempt to solve Problem A rests upon the following

Problem B. Find examples of admissible wavelets (both for $\alpha \neq 0$ and $\alpha = 0$) and compute c_α .

Now, let us try to prove (8.29). We say “try”, because along the way, we come across one more open problem related to application of the Fubini theorem; cf. justification of interchange of the order of integration in the proof of Theorem 2.2.

By (8.26) and (8.19),

$$\begin{aligned} \mathcal{H}I^\alpha f(x, a) &= \int_{\mathcal{P}_m} H_{a^{1/2}\eta a^{1/2}} I^\alpha f(x) w(\eta) d\eta \\ &= \int_{\mathcal{P}_m} I_-^{\alpha/2} [H_{(\cdot)} f(x)] (a^{1/2}\eta a^{1/2}) w(\eta) d\eta. \end{aligned}$$

Assume that x is fixed and denote $\psi(s) = H_s f(x)$. Then

$$\begin{aligned} \mathcal{H}I^\alpha f(x, a) &= \int_{\mathcal{P}_m} w(\eta) I_-^{\alpha/2} \psi(a^{1/2}\eta a^{1/2}) d\eta \\ &= \frac{1}{\Gamma_m(\alpha/2)} \int_{\mathcal{P}_m} w(\eta) d\eta \int_{a^{1/2}\eta a^{1/2}}^\infty \psi(s) |s - a^{1/2}\eta a^{1/2}|^{\alpha/2-d} ds \\ &= \frac{1}{\Gamma_m(\alpha/2)} \int_{\mathcal{P}_m} \psi(s) ds \int_0^{a^{-1/2}\eta a^{-1/2}} w(\eta) |s - a^{1/2}\eta a^{1/2}|^{\alpha/2-d} d\eta \\ &= |a|^{\alpha/2-d} \int_{\mathcal{P}_m} \psi(s) I_+^{\alpha/2} w(a^{-1/2} s a^{-1/2}) ds. \end{aligned}$$

Hence, the left-hand side of (8.29) transforms as follows.

$$\begin{aligned} I_\varepsilon f(x) &= \int_{\varepsilon I_m}^\infty \frac{da}{|a|^{m+1}} \int_{\mathcal{P}_m} \psi(s) I_+^{\alpha/2} w(a^{-1/2} s a^{-1/2}) ds \\ &= \int_{\mathcal{P}_m} \psi(s) \int_{\varepsilon I_m}^\infty I_+^{\alpha/2} w(a^{-1/2} s a^{-1/2}) \frac{da}{|a|^{m+1}} \quad (\text{set } a = \tau^{-1}) \\ &= \int_{\mathcal{P}_m} \psi(s) \int_0^{\varepsilon^{-1} I_m} I_+^{\alpha/2} w(\tau^{1/2} s \tau^{1/2}) d\tau \\ &= \varepsilon^{md} \int_{\mathcal{P}_m} \psi(\varepsilon s) dy \int_0^{\varepsilon^{-1} I_m} I_+^{\alpha/2} w(\tau^{1/2} \varepsilon^{1/2} s \varepsilon^{1/2} \tau^{1/2}) d\tau. \end{aligned}$$

Thus we have

$$(8.32) \quad I_\varepsilon f(x) = \int_{\mathcal{P}_m} \psi(\varepsilon s) k(s) ds = \int_{\mathcal{P}_m} H_{\varepsilon s} f(x) k(s) ds,$$

where

$$k(s) = \int_0^{I_m} I_+^{\alpha/2} w(\lambda^{1/2} s \lambda^{1/2}) d\lambda.$$

To get (8.29), it remains to show that $k(s)$ coincides with the function (8.30). We have

$$\begin{aligned}
k(s) &= \frac{1}{\Gamma_m(\alpha/2)} \int_0^{I_m} d\lambda \int_0^{\lambda^{1/2} s \lambda^{1/2}} w(s) |\lambda^{1/2} s \lambda^{1/2} - s|^{\alpha/2-d} ds \\
&\quad (\text{set } s = \lambda^{1/2} z \lambda^{1/2} \text{ and note that } w(\lambda^{1/2} z \lambda^{1/2}) = w(z^{1/2} \lambda z^{1/2})) \\
&= \frac{1}{\Gamma_m(\alpha/2)} \int_0^{I_m} |\lambda|^{\alpha/2} d\lambda \int_0^s |s - z|^{\alpha/2-d} w(z^{1/2} \lambda z^{1/2}) dz \\
&= \frac{1}{\Gamma_m(\alpha/2)} \int_0^s |s - z|^{\alpha/2-d} dz \int_0^{I_m} |\lambda|^{\alpha/2} w(z^{1/2} \lambda z^{1/2}) d\lambda \\
&= \frac{1}{\Gamma_m(\alpha/2)} \int_0^s |s - z|^{\alpha/2-d} \frac{dz}{|z|^{\alpha/2+d}} \int_0^z w(b) |b|^{\alpha/2} db \\
&= \frac{1}{\Gamma_m(\alpha/2)} \int_0^s w(b) |b|^{\alpha/2} u(b, s) db,
\end{aligned}$$

where

$$\begin{aligned}
u(b, s) &= \int_b^s |s - z|^{\alpha/2-d} \frac{dz}{|z|^{\alpha/2+d}} \quad (\text{set } z = r^{-1}) \\
&= \int_{s^{-1}}^{b^{-1}} |sr - I_m|^{\alpha/2-d} dr = |s|^{\alpha/2-d} \int_{s^{-1}}^{b^{-1}} |r - s^{-1}|^{\alpha/2-d} dr.
\end{aligned}$$

The last integral can be easily computed using the well-known formula for Siegel Beta functions

$$(8.33) \quad \int_a^b |r - a|^{\alpha-d} |b - r|^{\beta-d} dr = B_m(\alpha, \beta) |b - a|^{\alpha+\beta-d}$$

(many such formulas can be found, e.g., in [OR2]), and we have

$$(8.34) \quad u(b, s) = B_m(\alpha/2, d) \frac{|s - b|^{\alpha/2}}{|s|^d |b|^{\alpha/2}}, \quad B_m(\alpha/2, d) = \frac{\Gamma_m(\alpha/2) \Gamma_m(d)}{\Gamma_m(\alpha/2 + d)}.$$

Finally, we get

$$k(s) = \frac{\Gamma_m(d)}{|s|^d \Gamma_m(\alpha/2 + d)} \int_0^s w(b) |s - b|^{\alpha/2} ds = \frac{\Gamma_m(d)}{|s|^d} I_+^{\alpha/2+d} w(s) = \Lambda_{\alpha/2}(s).$$

Problem C. Although all calculations above go through smoothly, interchange of the order of integration remains unjustified. We do not know how to justify it and what additional requirements on the wavelet w should be imposed (if any). One of the obstacles is that $\int_0^\infty \neq \int_0^s + \int_s^\infty$, when we integrate over the higher-rank cone.

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